INTERRELATIONS BETWEEN THE SPECTRUM OF LESLIE MATRIX AND THE AGE PATTERN OF DEMOGRAPHIC POTENTIALS

Dalkhat M. Ediev (dalkhat@hotmail.com)

Karachay-Cherkessian state Technological Institute

It is shown that latent roots of population projection matrix–except for Lotka's coefficient–are uniquely determined by the age pattern of demographic potentials and the mortality regimen. Relative values of latent roots are uniquely determined by expected future demographic potentials of newborns. For arbitrary patterns of demographic potentials and mortality an instability circle is introduced, such that the ergodic property can be proved for the population reproduction model if and only if the intrinsic growth rate doesn't fall within the circle mentioned. Results obtained develop the stable population theory, the model of demographic potential's reproduction, and can be applied in constructing population projection models.

Introduction

The article deals with spectrum of population projection matrix in the model of demographic potentials reproduction [1], which generalizes the well-known class of cohort-component models [2, 3]. The following matrix relation describes population dynamic in the cohort-component model:

$$\mathbf{n}(t+1)=\mathbf{L}\mathbf{n}(t) \tag{1}$$

Here $\mathbf{n}(t)$ - is a vector containing the population sizes in different age groups, and $\mathbf{L}$ – is a population projection matrix (Leslie matrix):

$$\mathbf{n}(t)=\begin{pmatrix} n_0(t) \\ n_1(t) \\ \dots \\ \dots \\ n_X(t) \end{pmatrix}; \qquad \mathbf{L}=\begin{pmatrix} F_0 & F_1 & \dots & F_{X-1} & F_X \\ P_0 & 0 & & 0 & 0 \\ 0 & P_1 & & 0 & 0 \\ & & \dots & & \\ 0 & 0 & & P_{X-1} & 0 \end{pmatrix} \tag{2}$$

where $n_x(t)$ - is a size of x-th age group (usually– population of x to $x+1$ years old) at time t; "0" index corresponds to the youngest and "X" index corresponds to the oldest age groups; $P_x, x=\overline{0,X}$ - are the probabilities that person of x-th age group at time t will be alive in $x+1$-th age group at time $t+1$; $F_x, x=\overline{0,X}$ are age specific fertility coefficients corrected for infant mortality.

As time tends to infinity, population asymptotically converges to the stable equivalent population, which is independent of initial age structure and is determined by the reproduction coefficients of the cohort-component model [4]; the theory of stable populations dates back to the works of L. Euler, T. Malthus, and A. Lotka. The property of convergence to stability mentioned above (ergodic property) was examined for continuous models by Lotka, Sharp, and Coale; for discrete models – by Leslie, Lopez, and Parlett; and for stochastic models – by Cohen. Proof and bibliography can be found in Arthur [5, 6]. It is interesting that under the appropriate choice of distance to stable equivalent, the convergence to stability happens monotonously [7, 8].

All the works mentioned above were limited to models with nonnegative Leslie matrices however. In addition to mathematical advantages this choice is supported by the fact that surviving probabilities and fertility coefficients can't be negative. Yet, as earlier as in the very beginning works on population reproduction models [2, 9] it was noted that models with possibly negative coefficients in the first row of Leslie matrix could be of importance. In particular, negative coefficients arise in models of reproduction of some biological populations with "anti-reproductive" behavior of specimens of particular age groups (eggs and youngsters destruction, etc.) Similar situation can arise due to competition for resources between youngsters and specimens of other age groups. The model of demographic potential's reproduction also can lead to the population projection matrix with negative coefficients. This model, being a generalization of the cohort-component model, can be written in the (1)-(2) form. Coefficients $F_x, x=\overline{0,X}$ of first row of the projection matrix $\mathbf{L}$, however, aren't calculated on the base of fertility coefficients. Rather, they are based on mortality regime, reproduction rate and demographic potentials' age pattern [1]:

$$F_x = \lambda \frac{c_x}{c_0} - P_x \frac{c_{x+1}}{c_0} \qquad (3)$$

here $c_x, x=\overline{0,X}$ are age-specific demographic potentials, and λ is an intrinsic reproduction coefficient, which is an analogous to Lotka's coefficient [10] in cohort-component model; it determines the asymptotic rate of population increase. Demographic potential [1, 11] can be treated as a generalization of R.A. Fisher's reproductive value [12]. Coefficients (3) turn to common coefficients F_x of population projection matrix under appropriate choice of demographic potentials. The model of reproduction of demographic potential has many advantages, among them is a possibility to develop the model in aggregate form, without age-sex differentiation. Appropriate model were presented earlier and proved their suitability for practical needs [1, 13, 14, 15].

It is important to explore, whether the ergodic property, which was proved for cohort-component models with nonnegative coefficients, holds for more general class of models of demographic potential's reproduction. Earlier [4] it was shown that ergodicity holds for models mentioned if the projection matrix $\mathbf{L}$ meets requirements usually imposed on population projection matrices in cohort-component model. Nonnegativity of matrix coefficients, in particular, is to be met. Approaches developed, however, failed to work in case of projection matrices with possibly negative coefficients. In this work an attempt is made to develop general results on asymptotic behavior of models of reproduction of demographic potential, without imposing restrictions on signs of projection matrix's coefficients. Investigation of spectral properties of the projection matrix $\mathbf{L}$ turned out to be very helpful in this attempt.

1. Spectrum of a projection matrix and the ergodic property

It is widely known [2] that Lotka's coefficient λ is the only positive latent root of the population projection matrix, age structure of stable equivalent population is an appropriate right latent vector, and age structure of Fisher's reproductive value is an appropriate left latent vector. Similarly, for the model of demographic potential's reproduction, the reproduction coefficient λ is a latent root of the projection matrix and age patterns of stable equivalent population and of demographic potential are the right and left eigenvectors corresponding to the eigenvalue λ. In particular,

$$\begin{pmatrix} c_0 & c_1 & \dots & \dots & c_X \end{pmatrix} \begin{pmatrix} F_0 & F_1 & \dots & F_{X-1} & F_X \\ P_0 & 0 & & 0 & 0 \\ 0 & P_1 & & 0 & 0 \\ & & \dots & & \\ 0 & 0 & & P_{X-1} & 0 \end{pmatrix} = \lambda \begin{pmatrix} c_0 & c_1 & \dots & \dots & c_X \end{pmatrix} \qquad (4)$$

This expression can be easily obtained taking into account (3):

$$c_0 F_x + c_{x+1} P_x = c_0 \left(\lambda \frac{c_x}{c_0} - P_x \frac{c_{x+1}}{c_0} \right) + c_{x+1} P_x = \lambda c_x$$

It is known from the matrix theory [16: (lemma 8.2.7)] that if some nonzero latent root λ of matrix L is the only one, which has maximal absolute value among all the latent roots, then

$$\left(\frac{1}{\lambda} L \right)^t = \frac{\mathbf{n}^* \mathbf{c}^T}{\mathbf{n}^{*T} \mathbf{c}} + \left(\frac{1}{\lambda} L - \frac{\mathbf{n}^* \mathbf{c}^T}{\mathbf{n}^{*T} \mathbf{c}} \right)^t \xrightarrow[t \to \infty]{} \frac{\mathbf{n}^* \mathbf{c}^T}{\mathbf{n}^{*T} \mathbf{c}}$$

here $\mathbf{c}$ and $\mathbf{n}^*$ are left and right eigenvectors that correspond to the latent root mentioned. Hence, under the conditions mentioned, we have:

$$\frac{\mathbf{n}(t)}{\lambda^t}=\left(\frac{1}{\lambda}L\right)^t\mathbf{n}(0)\xrightarrow[t\to\infty]{}\frac{\mathbf{n}^*\mathbf{c}^T}{\mathbf{n}^{*T}\mathbf{c}}\mathbf{n}(0)=\mathbf{n}^*\frac{\mathbf{c}^T\mathbf{n}(0)}{\mathbf{n}^{*T}\mathbf{c}}, \tag{5}$$

i.e. population's age structure converges asymptotically to the structure of stable equivalent population. In other words, the ergodic property is assured for the reproduction model.

Hence, the ergodic property holds for linear reproduction model if and only if the reproduction coefficient (Lotka's coefficient) is the only latent root with maximal absolute value. Sufficiency in the proposition made follows from (5), and necessity follows from the fact that if latent root exists, which has a modulus more or equal to that of the reproduction coefficient, then convergence to stable population can be guaranteed only for initial population structures, which are orthogonal to left eigenvector corresponding to the reproduction coefficient. Last statement follows from the orthogonality of left and right eigenvectors corresponding to different eigenvalues (stable population's structure, in particular, is orthogonal to all the left eigenvectors, which correspond to latent roots other than the reproduction coefficient). Hence, under the ergodic property, following condition is to be satisfied by any left eigenvector $\mathbf{c}^{(k)}$, which corresponds to latent root $\lambda^{(k)}$ differing from the reproduction coefficient λ:

$$\frac{1}{\lambda^t}C^{(k)}(t)\overset{def}{=}\frac{1}{\lambda^t}\sum_{x=0}^{X}c^{(k)}{}_x n_x(t)=\frac{1}{\lambda^t}\mathbf{c}^{(k)T}\mathbf{n}(t)=\frac{1}{\lambda^t}\mathbf{c}^{(k)T}L^t\mathbf{n}(0)=\left(\frac{\lambda^{(k)}}{\lambda}\right)^t\mathbf{c}^{(k)T}\mathbf{n}(0)\xrightarrow[t\to\infty]{}0, \tag{6}$$

But this condition can be proved if and only if $\left|\lambda^{(k)}\right|<|\lambda|$ or $\mathbf{c}^{(k)}$ is orthogonal to initial population structure $\mathbf{n}(0)$. Meanwhile, in arbitrarily small neighborhood of the stable population's age structure such a population structures can be found, which aren't orthogonal to the left eigenvector mentioned. I.e. the ergodic property can't be proved even for initial populations from arbitrarily small subset of population structures, if the reproduction coefficient isn't the only latent root with maximal absolute value from the spectrum of projection matrix.

2. Spectrum of Leslie matrix and demographic potentials age pattern

Projection matrix has many latent roots and vectors, which differ from the reproduction coefficient λ and from the structures of stable population and demographic potentials. Let write the secular equation in order to explore these latent roots and vectors:

$$\det(\mathbf{L}-\mu\mathbf{I})=\det\begin{pmatrix} F_0-\mu & F_1 & \dots & F_{X-1} & F_X \\ P_0 & -\mu & & 0 & 0 \\ 0 & P_1 & & 0 & 0 \\ & & \dots & -\mu & \\ 0 & 0 & & P_{X-1} & -\mu \end{pmatrix}=0, \text{ i.e.}$$

$$\sum_{x=0}^{X}L_xF_x\mu^{-x}=L_0\mu, \tag{7}$$

where $L_x=L_0P_0P_1...P_{x-1}, x=\overline{0,X}$ are stationary population's numbers, which being divided by each other correspond to probabilities of surviving from one age group to another.

Expression (7), in particular, is satisfied by the reproduction coefficient λ: $\sum_{x=0}^{X}L_xF_x\lambda^{-x}=L_0\lambda$. Substituting the expression for F_x into (7) we have:

$$\sum_{x=0}^{X}L_x\left(\lambda\frac{c_x}{c_0}-P_x\frac{c_{x+1}}{c_0}\right)\mu^{-x}=L_0\mu$$

$$\lambda\sum_{x=0}^{X}L_xc_x\mu^{-x}-\sum_{x=0}^{X}L_{x+1}c_{x+1}\mu^{-x}=L_0c_0\mu$$

$$\lambda\sum_{x=0}^{X}L_xc_x\mu^{-x}-\mu\sum_{x=1}^{X}L_xc_x\mu^{-x}-\mu L_0c_0=0$$

$$(\lambda-\mu)\sum_{x=0}^{X}L_x c_x \mu^{-x}=0$$

I.e. the reproduction coefficient is one of roots of the secular equation and the rest of the spectrum can be obtained solving the following equation:

$$\sum_{x=0}^{X}L_x c_x \mu^{-x}=0 \qquad (8)$$

This expression implies, particularly, that given the nonnegativity of demographic potentials c_x, which aren't equal to zero for all the ages with positive stationary population numbers L_x, the reproduction coefficient is the only positive latent root of the projection matrix. Other latent roots are either negative, or can be arranged in conjugate complex pairs.

3. The circle of instability of a demographic potential's reproduction model

Another important conclusion, which can be made based on the (8) is that secular equation's roots other than the reproduction coefficient are uniquely determined by regimen of mortality and by age pattern of demographic potentials, and *aren't directly dependent* on the value of the reproduction coefficient. Hence, we have

Theorem *(on the instability circle of a model of demographic potential's reproduction).* Let the age pattern $c_x, x=\overline{0,X}$ of demographic potentials and stationary population numbers $L_x, x=\overline{0,X}$ are specified. Then the circle $\Lambda=\{\lambda:|\lambda|\le\lambda^{\min}\}$ exists, such that the reproduction model possesses the ergodic property if and only if the reproduction coefficient doesn't belong to that circle. The circle mentioned will be called as an *instability circle*, and its radius will be called as an *instability radius*.

Remark 1. Indeed, the instability radius introduced in the theorem equals to the maximal modulus of roots of equation (8), i.e. $\lambda^{\min}=\max\left\{|\mu|:\sum_{x=0}^{X}L_x c_x \mu^{-x}=0\right\}$.

Remark 2. It follows from the theorem that for any pattern of demographic potential the ergodic property will take place under appropriate choice of the true reproduction rate. Models with negative coefficients in Leslie matrix or even with negative demographic potentials can also possess the ergodic property. These models weren't investigated in the literature, and results on asymptotic properties of demographic models, which were obtained earlier, can't be applied to the models mentioned.

Remark 3. Another consequence from the theorem is that if the ergodic property holds for the demographic potential's reproduction model with some value of true reproduction rate, then ergodicity holds for all models with higher levels of true reproduction rate as well.

Let us illustrate results obtained on the model based on US 1991 females' reproductive values and survivorship rates (appropriate models were used by author in previous works). Fig. 1 presents the spectrum of appropriate projection matrix. The figure depicts the distribution of the latent roots in complex plane (horizontal axis corresponds to real parts and vertical axis – to imaginary parts of latent roots). There are two circles shown on the fig. 1. First circle (black) is of unity radius, and another (green) has the radius of the instability circle (it passes through the latent roots having the next modulus after the reproduction coefficient). The fact that the reproduction coefficient (it is the most righter point of the spectrum) lies within the unity circle points to the depopulating regime of reproduction. And the fact that the reproduction rate lies outside the instability circle points to the ergodic property. The latent root having the next modulus after the reproduction rate has the absolute value of 0,9624, i.e. under the demographic potentials and mortality of the US 1991 female population ergodicity can be guaranteed if and only if the intrinsic growth rate is more than -3,76% annually. It worth to note that the complex pair of latent roots having the modulus equal to the instability radius correspond to attenuating oscillations with the period of 28 years, i.e. to so called demographic waves. Under intrinsic rates less than -3,76% these waves will become amplifying ones and that will break the ergodicity.

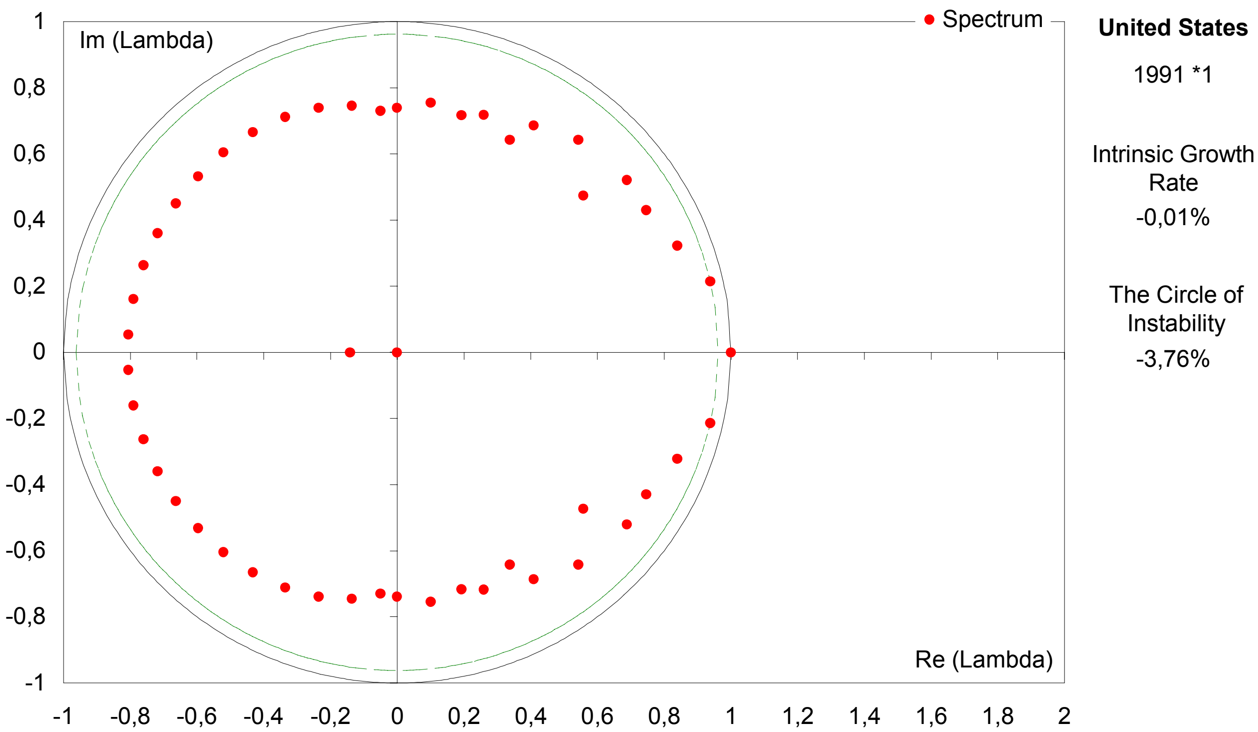

Fig. 1. Spectrum of the projection matrix based on US 1991 vital data (females).

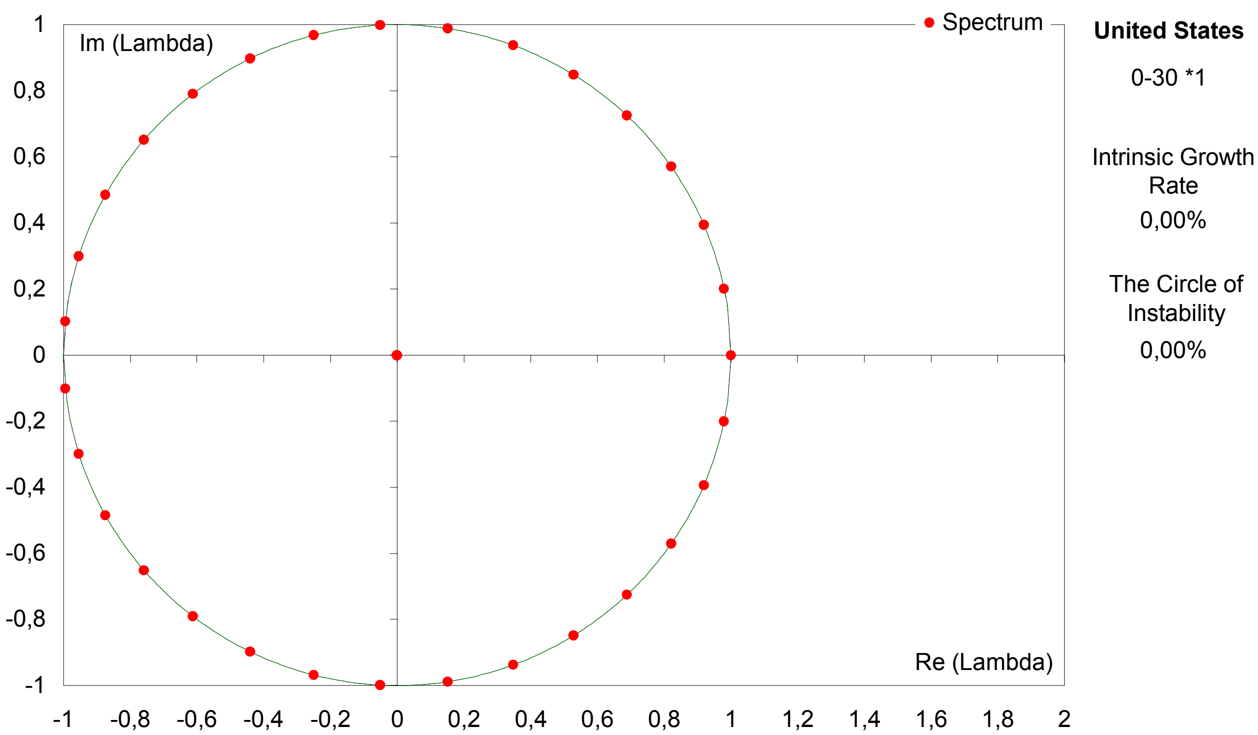

Fig. 2. Spectrum of the projection matrix based on US 1991 mortality data (females) and on the demographic potential taken equal to size of the 0-30 age group (zero intrinsic growth rate).

Figure 2 also presents spectrum of the model based on mortality pattern of US 1991 female population, but the size of 0-30 age segment is used as a demographic potential in spite of direct use of Fisher's reproductive values (Lotka's coefficient is chosen to be one). Rational under this choice is given by results of several works [7, 17-21], which stated that the segment mentioned determines the asymptote of the population dynamics. Results of population forecasting using the appropriate reproduction model were unstable, however [1]. The spectrum presented on figure 2 explains this instability – instability radius equals to 0%, i.e. even the model with zero growth doesn't possess the ergodic property. From the figure 2 it can also be noted that models based on the reproduction of 0-30 age segment could be useful for rapidly growing populations.

Next spectrum (Fig. 3) corresponds to the model with high fertility, which is based on US 1960 data (the middle of the baby-boom). Relatively small instability circle (–1,92%) points to the fact that reproductive values calculated for population with high fertility are good enough to build models with low Lotka's coefficients, so that the ergodic property remains true.

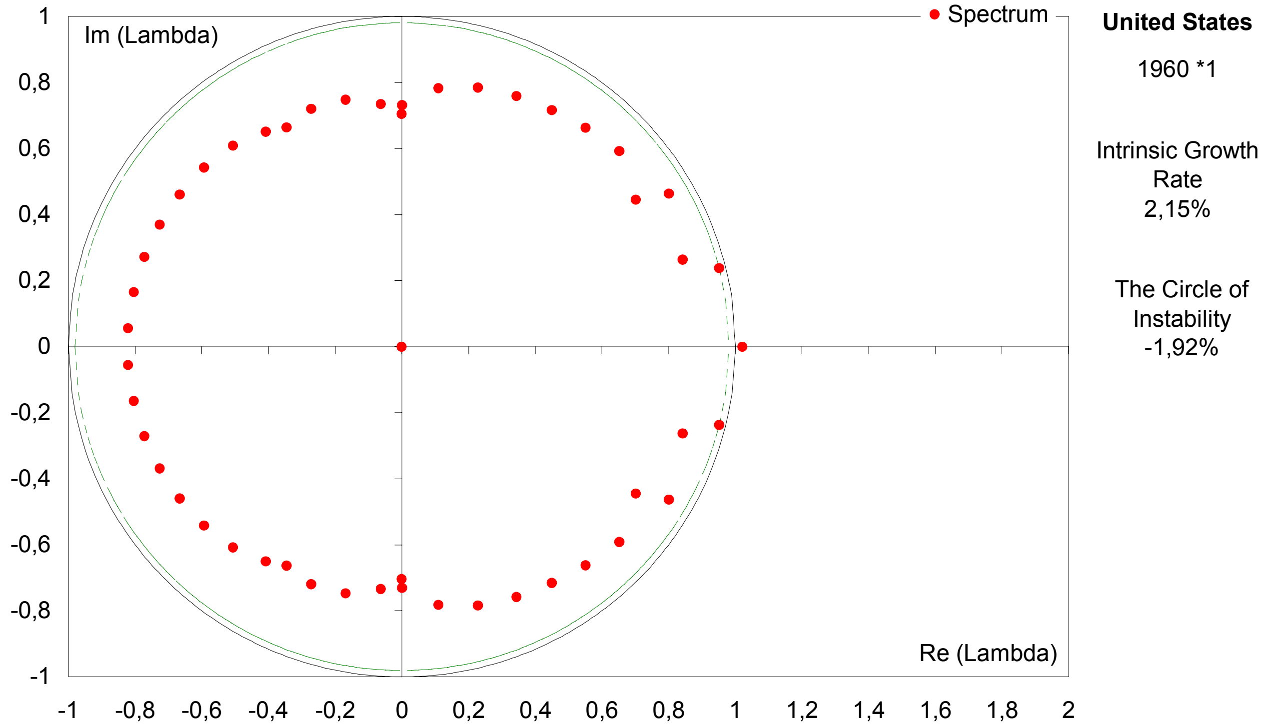

Fig. 3. Spectrum of the projection matrix based on US 1960 vital data (females).

It is useful to rewrite equation (8) in terms of expected future potentials of newborns, $u_x \overset{def}{=} \frac{c_x L_x \lambda^{-x}}{c_0 L_0}$, in order to explore ratios of latent roots to the reproduction coefficient (they are these ratios, which determine whether or not the ergodicity takes place):

$$0 = \sum_{x=0}^{X} L_x c_x \mu^{-x} = \sum_{x=0}^{X} u_x \lambda^x c_0 L_0 \mu^{-x} = c_0 L_0 \sum_{x=0}^{X} u_x \left(\frac{\mu}{\lambda}\right)^{-x}, \text{ i.e.}$$

$$\sum_{x=0}^{X} u_x \left(\frac{\mu}{\lambda}\right)^{-x} = 0 \qquad (9)$$

Hence, the relative latent roots $\frac{\mu}{\lambda}$ of the projection matrix are uniquely determined by expected future potentials u_x and, given the values of u_x, are independent on mortality, the reproduction coefficient, or demographic potentials' age pattern. From this statement we have the following

Theorem *(On the Ergodicity in a Model of Demographic Potential's Reproduction)*. Age pattern $u_x, x = \overline{0, X}$ of expected future demographic potentials of newborn uniquely determines proportions between the latent roots of the projection matrix and, therefore, whether or not the demographic potential's reproduction model possesses the ergodic property.

Remark. This theorem introduces a simple way of choosing a pattern of demographic potentials that guaranties the ergodicity. The pattern of demographic potentials should be based on appropriate structure of $u_x, x = \overline{0, X}$, on given mortality regime, and on the lowest possible value for the reproduction coefficient λ.

Expressions similar to (4), (8), and (9) could also be written for other latent roots and vectors, i.e. any latent root of the projection matrix and appropriate latent vector together with the mortality regime uniquely determine the first row of the projection matrix and the entire spectrum of this matrix.

4. Dependency of demographic potentials' age pattern on projection matrix's spectrum

Following result could be of importance in constructing the reproduction model as well as in finding the spectrum of the projection matrix. From (8), (9) it is seen, that spectrum of the projection matrix is determined by the age pattern of demographic potentials. Vice versa, it follows from the same equations that the spectrum determines the age pattern of expected future demographic potentials and – given the mortality regimen – the age pattern of demographic potentials. Indeed, given the known spectrum $\lambda_k, k=\overline{0,X}$, $\lambda_0 \overset{def}{=} \lambda$, of the projection matrix, expression (9) could be rewritten as:

$$0=\sum_{x=0}^{X} u_x\left(\frac{\lambda}{\mu}\right)^x=\prod_{k=1}^{X}\left(1-\frac{\lambda_k}{\mu}\right) \tag{10}$$

Hence, expanding the right side and equating the summands with corresponding powers of μ, we have:

$$u_x=\frac{1}{x!}\sum_{\substack{k1,k2,\dots,kx=1\\ k1\neq k2,k1\neq k3,\dots,(kx-1)\neq kx}}^{X}\frac{-\lambda_{k1}}{\lambda}\frac{-\lambda_{k2}}{\lambda}\cdots\frac{-\lambda_{kx}}{\lambda}, x=\overline{0,X} \tag{11}$$

Or, equivalently:

$$u_x=\sum_{k1=1}^{X}\sum_{k2=k1+1}^{X}\sum_{k3=k2+1}^{X}\cdots\sum_{kx=(kx-1)+1}^{X}\frac{-\lambda_{k1}}{\lambda}\frac{-\lambda_{k2}}{\lambda}\cdots\frac{-\lambda_{kx}}{\lambda}, x=\overline{0,X} \tag{12}$$

Expressions (11), (12) can be transformed for practical use in a following way. First, let us introduce short notations:

$$\Sigma^x=\sum_{\substack{k1,k2,\dots,kx=1\\ k1\neq k2,k1\neq k3,\dots,(kx-1)\neq kx}}^{X}\frac{-\lambda_{k1}}{\lambda}\frac{-\lambda_{k2}}{\lambda}\cdots\frac{-\lambda_{kx}}{\lambda}, \qquad x=\overline{0,X}, \tag{13}$$

i.e. $\Sigma^x, x=\overline{0,X}$ is a sum of all possible products of x different multipliers $\frac{-\lambda_j}{\lambda}$. Equation (11) appears in new notations as:

$$u_x=\frac{1}{x!}\Sigma^x, x=\overline{0,X} \tag{14}$$

Furthermore, let us denote by $\Sigma^x_{\neq i}, x,i=\overline{0,X}$ the sum of (13) type, which doesn't contain the multiplier $\frac{-\lambda_i}{\lambda}$ in any summand. Then we have:

$$u_x=\frac{1}{x!}\Sigma^x=\frac{1}{x!}\sum_{i=1}^{X}\left(-\frac{\lambda_i}{\lambda}\right)\Sigma^{x-1}_{\neq i}=\frac{1}{x!}\sum_{i=1}^{X}\left(-\frac{\lambda_i}{\lambda}\right)\left(\Sigma^{x-1}-(x-1)\left(-\frac{\lambda_i}{\lambda}\right)\Sigma^{x-2}_{\neq i}\right)=$$

$$=\frac{1}{x!}\sum_{i=1}^{X}\left(-\frac{\lambda_i}{\lambda}\right)\left(\Sigma^{x-1}-(x-1)\left(-\frac{\lambda_i}{\lambda}\right)\left(\Sigma^{x-2}-(x-2)\left(-\frac{\lambda_i}{\lambda}\right)\Sigma^{x-3}_{\neq i}\right)\right)=\dots$$

$$=-\frac{1}{x}\sum_{i=1}^{X}\sum_{k=1}^{x}\left(\frac{\lambda_i}{\lambda}\right)^k\frac{\Sigma^{x-k}}{(x-k)!}=-\frac{1}{x}\sum_{i=1}^{X}\sum_{k=1}^{x}\left(\frac{\lambda_i}{\lambda}\right)^k u_{x-k}=-\frac{1}{x}\sum_{y=0}^{x-1}u_y\sum_{i=1}^{X}\left(\frac{\lambda_i}{\lambda}\right)^{x-y}$$

I.e., we've proved the following recurrent relations for expected future demographic potentials under known spectrum of the projection matrix:

$$u_0=1,\ u_x=-\frac{1}{x}\sum_{y=0}^{x-1}u_y\sum_{i=1}^{X}\left(\frac{\lambda_i}{\lambda}\right)^{x-y}, x=\overline{1,X} \tag{15}$$

Following relations for demographic potentials can be obtained from expressions (15):

$$c_x = -\frac{1}{xL_x}\sum_{y=0}^{x-1} L_y c_y \sum_{i=1}^{X} \lambda_i^{x-y}, x = \overline{1, X} \qquad (16)$$

Using (15), (16) one could easy the calculation of the spectrum of projection matrix. The point is that these expressions contain in explicit form all the eigenvalues to be calculated. Hence, for example, all the eigenvalues could be found simultaneously by solving one of the following minimization problems:

$$Z\left(\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}, \ldots, \frac{\lambda_X}{\lambda}\right) = \sum_{x=1}^{X} (u_x - \hat{u}_x)^2 \xrightarrow[\{\lambda_k/\lambda\}]{} \min \qquad (17)$$

here $\hat{u}_x = \hat{u}_x\left(\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}, \ldots, \frac{\lambda_X}{\lambda}\right)$ are the future demographic potentials estimated by the means of (15); or

$$Z(\lambda_1, \lambda_2, \ldots, \lambda_X) = \sum_{x=1}^{X} (c_x - \hat{c}_x)^2 \xrightarrow[\{\lambda_k\}]{} \min \qquad (18)$$

here $\hat{c}_x = \hat{c}_x(\lambda_1, \lambda_2, \ldots, \lambda_X)$ are demographic potentials estimated by the means of (16).

They were these procedures, which were used to calculate the spectrums presented above. It worth to note that the latter procedure (18) was usually more effective compared to (17).

In addition to procedures mentioned, expressions (15), (16) can also be used to find sums of latent roots in different degrees when demographic potentials are known:

$$M^x \stackrel{def}{=} \sum_{i=1}^{X} \left(\frac{\lambda_i}{\lambda}\right)^x = -xu_x - M^1 u_{x-1} - M^2 u_{x-2} - \ldots - M^{x-1} u_1,\ x = \overline{1, X}, \text{ i.e.}$$

$$M^x = -xu_x - \sum_{y=1}^{x-1} M^y u_{x-y},\ x = \overline{1, X} \qquad (19)$$

After these values are obtained latent roots can be found by solving system of equations (19) with respect to $\frac{\lambda_i}{\lambda},\ i = \overline{1, X}$. It worth to note that if some of latent roots are obtained, than others can be found by using a new set of potentials and, respectively, a new projection matrix of smaller dimension. Indeed, if some of latent roots $\frac{\lambda_i}{\lambda}, i = \overline{1, \tau}$ are known, then we have for the rest of them:

$$M_{\tau+1}{}^x \stackrel{def}{=} \sum_{i=\tau+1}^{X} \left(\frac{\lambda_i}{\lambda}\right)^x = M^x - \sum_{i=1}^{\tau} \left(\frac{\lambda_i}{\lambda}\right)^x,\ x = \overline{1, X}$$

These new sums correspond to new set of potentials, to new projection matrix, and to new set of latent roots; latter being a latent roots of the initial projection matrix.

When the values of $M^x,\ x = \overline{1, X}$ are calculated according to the (19), one could build another minimization procedure to find the spectrum of latent roots:

$$Z\left(\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}, \ldots, \frac{\lambda_X}{\lambda}\right) = \sum_{x=1}^{X} \left(M^x - \hat{M}^x\right)^2 \xrightarrow[\{\lambda_k/\lambda\}]{} \min, \qquad (20)$$

here $\hat{M}^x = \sum_{i=1}^{X} \left(\frac{\lambda_i}{\lambda}\right)^x$. Although procedure (20) was of lesser effectiveness compared to (17) or (18), it is potentially useful for models with bigger dimension, as the ideas of dynamic programming can be used to solve the appropriate optimization problem.

Acknowledgements

Author is pleased to the Economic Education and Research Consortium for provided access to the online library JSTOR. The work was conducted within the project “Development of the demographic potential’s model and of its applications” supported by Russian Ministry of Education (2002 stage: “Exploring the asymptotic properties of the model of demographic potential’s reproduction”).

References

1. Ediev D.M. Application of the demographic potential concept to understanding the Russian population history and prospects: 1897-2100 // Demographic Research. 2001. Vol.4, Article 9. P. 289-336. http://www.demographic-research.org/volumes/vol4/9/4-9.pdf
2. Leslie P.H. On the use of matrices in certain population mathematics // Biometrika. 1945. XXXV. P. 183-212.
3. Modern Demography / Kvasha A.Y. and Iontcev V.A. , eds. Moscow, 1995. P. 159-163.
4. Keyfitz N. Age distribution and stable equivalent // Demography. 1969. № 6. P. 261-269.
5. Arthur W.B. Why a population converges to stability // American mathematical monthly. 1981. Vol. 88. № 8. P. 557-563.
6. Arthur W.B. The ergodic theorems of demography: a simple proof // Demography. 1982. № 19. P. 439-445.
7. Schoen R., Kim Y.J. Movement toward stability as a fundamental principle of population dynamics // Demography. 1991. № 28. P. 455-466.
8. Ediev D.M. On conditions of monotonic convergence of population structure to the structure of stable equivalent population in Kullback distance within the model of reproduction of demographic potential // Investigated in Russia (Demographic section), 2002, 17: 182-190. http://zhurnal.ape.relarn.ru/articles/2002/017.pdf
9. Chapman R.N. The causes of fluctuations of populations of insects // Proc. Hawaii Ent. Soc. 1933. 8, no. 2. P. 279-292.
10. Dublin L. I., Lotka A. J. On the true rate of natural increase as exemplified by the population of the United States, 1920 // Journal of American Statistical Association. 1925. №20. P. 305-339.
11. Ediev D.M. Demographic and economic-demographic potentials. Ph.D. thesis. Moscow: Moscow Institute of Physics and Technology, 1999.
12. Fisher R.A. The genetical theory of natural selection. N.-Y.: Dover Publications, 1930.
13. Ediev D.M. Principles of aggregate economic-demographic modeling based on demographic potentials' technique // Demographic-macroeconomic modeling. Rostock, 2000. 17 pp. www.demogr.mpg.de/Papers/workshops/ws001011.htm
14. Ediev D.M. Aggregate Population Forecasting With the Use of Demographic Potentials Technique // Investigated In Russia, 2001, 38e: 408-431. http://zhurnal.ape.relarn.ru/articles/2001/038e.pdf
15. Эдиев Д.М. Реконструкция показателей иммиграции в США с использованием модели демографического потенциала // Электронный журнал "Исследовано в России", 140, стр. 1601-1618, 2001 г. http://zhurnal.ape.relarn.ru/articles/2001/140.pdf
16. Horn R.A., Johnson Ch.R. Matrix analysis. Cambridge Univ. Press, 1986.
17. Preston S. H. The relation between actual and intrinsic growth rates // Population Studies. 1986. №40. P. 343-351.
18. Preston S. H. Coale A. J. Age structure growth, attrition, and accession: a new synthesis // Population Index. 1982. №48. P. 215-259.
19. Kim Y. J. Schoen R. Population momentum expresses population aging // Demography. 1997. №34. P. 421-427.
20. Kim Y. J. Schoen R. Sarma P. S. Momentum and the growth-free segment of a population // Demography. 1991. №28. P. 159-173.
21. Wachter K. W. Age group growth rates and population momentum // Population Studies. 1988. №42. P. 487-494.